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Fractal Measures and Mean p-Variations

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Recently Strichartz proved that if μ is locally uniformly α-dimensional on $\mathbb{R}^d$, then

$$\sup_{T\geqslant 1}\left(\frac{1}{T^{d-\alpha}}\int_{B_T}|(\mu_f)^\wedge|^2\right)^{1/2}\leqslant C_1\,\|f\|_{L^2(\mu)}\qquad\forall f\in L^2(\mu),$$

where $0\leqslant\alpha\leqslant d$, and B_T denotes the ball of radius T center at 0; if μ is self-similar and satisfies a certain open set condition, he also obtained a formula for the α so that $0<\limsup_{T\to\infty}(1/T^{d-\alpha})\int_{B_T}|(\mu_f)^\wedge|^2<\infty$. The α can serve, in some sense, as the dimensional index of the measure μ. By using the mean p-variation and the Tauberian theorems, we extend the first inequality and its variants to p, q forms, and give necessary and sufficient conditions on μ for such inequalities to hold; we then use the mean quadratic variation to study some self-similar measures μ on $\mathbb{R}$ which do *not* satisfy the open set condition: the μ's that are constructed from $S_1x=\rho x$, $S_2x=\rho x+(1-\rho)$, $1/2<\rho<1$ with weights 1/2 each. The index α for μ corresponding to $\rho=(\sqrt{5}-1)/2$ is calculated. The expression for such α is significantly different from the one obtained by Strichartz.

1. INTRODUCTION

Let $B_r(x)$ denote the unit ball of radius r with center at x, and write $B_r(0)$ as B_r for convenience. A positive σ-finite Borel measure μ on $\mathbb{R}^d$ is called *locally uniformly α-dimensional*, $0\leqslant\alpha\leqslant d$, if $\mu(B_r(x))\leqslant Cr^\alpha$ for all $0<r<1$, $x\in\mathbb{R}^d$. This class of measures was introduced by Strichartz [Str 1, Str 2] to study the Fourier transformation of fractal measures. He showed that if μ is such a measure, then there exists $C_1>0$ such that

$$\sup_{T\geqslant 1}\left(\frac{1}{T^{d-\alpha}}\int_{B_T}|(\mu_f)^\wedge|^2\right)^{1/2}\leqslant C_1\,\|f\|_{L^2(\mu)}\qquad\forall f\in L^2(\mu),\tag{1.1}$$

where $d\mu_f=f\,d\mu$. Moreover μ is absolutely continuous with respect to the α-Hausdorff measure ω_α(which is not σ-finite on $\mathbb{R}^d$), and has a decomposi-

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tion $\mu = \phi \, d\omega_\alpha + \nu$ where ν is *null* with respect to ω_α; if $\phi = \chi_E$ where E is a ω_α-regular subset of $\mathbb{R}^d$ (in this case, α is necessarily an integer [F1]), then there exists $C_2 > 0$ such that

$$\lim_{T \to \infty} \frac{1}{T^{d-\alpha}} \int_{B_T} |(\mu_f)^\wedge|^2 = C_2 \int_E |f|^2 \, d\omega_\alpha \qquad \forall f \in L^2(\mu). \tag{1.2}$$

Identity (1.2) generalizes simultaneously the following celebrated results:

(i) The Plancherel formula where μ is taken to be the Haar measure on $\mathbb{R}^d$ $(\alpha = d)$.

(ii) The Wiener identity for bounded measures μ on $\mathbb{R}^d$,

$$\lim_{T \to \infty} \frac{1}{T^d} \int_{B_T} |\hat{\mu}|^2 = C \sum_{x \in \mathbb{R}^d} |\mu\{x\}|^2 \tag{1.3}$$

$(\alpha = 0)$.

(iii) The identity of Agmon and Hörmander [AH], which takes the form (1.2) with μ a surface measure on a C^1-submanifold of $\mathbb{R}^d$ (α is an integer between 1 to d).

It also partially extends

(iv) The Besicotvich identity of almost periodic functions,

$$\lim_{T \to \infty} \frac{1}{T^d} \int_{B_T} |F|^2 = \sum_{n=1}^{\infty} |c_n|^2,$$

where $F(x) = \sum_{n=1}^{\infty} c_n \, e^{ia_n \cdot x}$, $a_n, x \in \mathbb{R}^d$, $c_n \in \mathbb{C}$.

Strichartz then used (1.1) and (1.2) to study the multipliers and the restriction theorems of $L^p(\mu)$ to $L^q(\mathbb{R}^d)$ [Str2], and in a sequence of papers following [Str3–Str5], he made further investigation of (1.2) for self-similar fractal measures, and also extended some results to Riemannian manifolds.

There is yet another well-known formula in this direction: The Wiener–Plancherel identity on $\mathbb{R}$ [W1],

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f|^2 = \lim_{h \to 0} \frac{1}{2h} \int_{-\infty}^{\infty} |\Delta_h W(f)|^2 \tag{1.4}$$

whenever either one limits exists, where $\Delta_h g(x) = g(x+h) - g(x+h)$, $h > 0$, and $W(f)$ is the Wiener transformation (integrated Fourier transformation) of f,

$$W(f)(x) = \int_{|t| \geq 1} \frac{f(t) \, e^{-2\pi i t}}{-2\pi i t} \, dt + \int_{|t| < 1} \frac{f(t)(e^{-2\pi i t} - 1)}{-2\pi i t} \, dt.$$

Recently the identity has been extended to $\mathbb{R}^d$ in [BBE, B, LW]. The related Banach spaces, dualities, isomorphisms, multipliers, and Hilbert transformations were studied in [CL1–CL3, H, L, LL].

By using Wiener's Tauberian theorem, it is not difficult to replace (1.4) by

$$\lim_{T\to\infty}\frac{1}{(2T)^{1-\alpha}}\int_{-T}^{T}|f|^2=\lim_{h\to 0}\frac{1}{(2h)^{1+\alpha}}\int_{-\infty}^{\infty}|\Delta_h W(f)|^2,$$

where $0\leqslant\alpha<1$. Note that if μ is a bounded Borel measure on $\mathbb{R}$, and if $f=\hat{\mu}$, then $W(f)=F+c$ a.e. where $F(x)=\mu(-\infty,x]$. Consequently we have

$$\lim_{T\to\infty}\frac{1}{(2T)^{1-\alpha}}\int_{-T}^{T}|\hat{\mu}|^2=\lim_{h\to 0}\frac{1}{(2h)^{1+\alpha}}\int_{-\infty}^{\infty}|\mu(x-h,x+h]|^2, \tag{1.5}$$

analogous to (1.3).

For a positive mekasure μ on $\mathbb{R}^d$, we will call

$$\limsup_{h\to 0}\frac{1}{(2h)^{d+\alpha}}\int_{\mathbb{R}^d}\mu(Q_h(x))^2\,dx$$

($Q_h(x)$ is the cube of size $(2h)^d$, centered at x) *upper α-mean quadratic variation* (m.q.v.) of μ. If the above limit exists, we simply call it the α-m.q.v. The *m.q.v. index* α of μ is defined to be

$$\inf\left\{\alpha: 0<\limsup_{h\to 0}\frac{1}{(2h)^{d+\alpha}}\int_{\mathbb{R}^d}\mu(Q_h(x))^2\,dx\right\}.$$

Note that the above set is nonempty, it always contains $\alpha=d$. (For otherwise, the zero of the limit supremum as $h\to 0$ implies that

$$\sup_{h>0}\frac{1}{(2h)^{2d}}\int_{\mathbb{R}^d}\mu(Q_h(x))^2\,dx<\infty.$$

By [HL], μ is absolutely continuous with $d\mu/dx=g$ in $L^2(\mathbb{R})$ and

$$0=\limsup_{h\to 0}\frac{1}{(2h)^{2d}}\int_{\mathbb{R}^d}\mu(Q_h(x))^2\,dx=\int_{\mathbb{R}^d}g^2.$$

Hence $\mu=0$ and is a contradiction.) The index α can serve, in some sense, as the dimension of the measure μ.

For the proof of (1.1) and (1.2) in [Str2], and also in [Str3–Str5], the technique depends heavily on the evaluation of the Gaussian kernel in order to get hold of the locally uniformly α-dimensional property of μ and its Fourier transformation. Identity (1.5) reveals such a relationship more explicitly. Our goal in this paper is to make use of the m.q.v. (and more general, the mean p-variation) to investigate the fractal measures. One of the major results is to prove, for $1 \leqslant p \leqslant q \leqslant \infty$, a necessary and sufficient condition of μ on $\mathbb{R}^d$ for the inequality

$$\sup_{0<h\leqslant 1} \left(\frac{1}{(2h)^{d+\alpha(p-1)}} \int_{\mathbb{R}^d} |\mu_f| \, (Q_h(u))^p \, du \right)^{1/p} \leqslant C \, \|f\|_{L^q(\mu)} \qquad \forall f \in L^q(\mu) \tag{1.6}$$

to hold (Theorem 2.3). By using a special type of Tauberian theorem, we can reduce the above for $1 \leqslant p \leqslant 2$, $p \leqslant q \leqslant \infty$, to

$$\sup_{T\geqslant 1} \left(\frac{1}{T^{d-\alpha}} \int_{B_T} |(\mu_f)^\wedge|^{p'} \right)^{1/p'} \leqslant C' \, \|f\|_{L^q(\mu)} \qquad \forall f \in L^q(\mu). \qquad (1.1)'$$

(Theorem 3.5). In particular for $p = q = 2$, the condition on μ reduces to Strichartz's condition of locally uniform α-dimension. The above inequalities can also be extended to the case of lim sup (Theorems 2.8, 3.8).

Recall that a regular Borel measure μ on $\mathbb{R}^d$ is called a *self-similar measure* [H] if μ is a probability measure and satisfies

$$\mu = \sum_{j=1}^{m} a_j \mu \circ S_j^{-1},$$

where $S_j(x) = \rho_j R_j x + b_j$ with $0 < \rho_j < 1$, R_j rotations on $\mathbb{R}^d$, and $b_j \in \mathbb{R}^d$, $j = 1, ..., m$. Strichartz [Str4] investigated such μ with the $\{S_j\}_{j=1}^m$ satisfying the "strong open set condition," and determined the dimensional index α of μ explicitly. An improvement of his result is given in [LW]. Specifically if α is such that

$$\sum_{j=1}^{m} a_j^2 \rho_j^{-\alpha} = 1, \tag{1.7}$$

then

$$\frac{1}{T^{d-\alpha}} \int_{B_T} |\hat{\mu}|^2 = p(T) + E(T), \tag{1.8}$$

where $\lim_{T\to\infty} E(T) = 0$, and $p(\lambda T) = p(T) \not\equiv 0$, or $p \equiv \text{constant} \neq 0$ according to $\{-\ln \rho_j\}_{j=1}^n$ is arithmetic or non-arithmetic. In the first case $\ln \lambda$, $\lambda > 1$, is the g.c.d. of $\{-\ln \rho_j\}_{j=1}^n$. Note that if $a_j, j = 1, ..., m$, are the nature weights (i.e., $a_j = \rho_j^{-\alpha}$), then α equals the dimension of the self-similar set induced by the similarities $\{S_j\}_{j=1}^m$.

In the second part of the paper we make an attempt to study the self-similar measures which do *not* satisfy the open set condition; we consider self-similar measures μ on $\mathbb{R}$ with $\rho_1=\rho_2=\rho$, $1/2<\rho<1$, and $a_1=a_2=1/2$. The situation is more complicated than the previous case (where the corresponding ρ is between 0 and 1/2). The measure μ can be identified, up to a scaling and a homothetic translation, with the distribution function F of the random variable $X=\sum_{n=1}^{\infty}\rho^n X_n$ where $\{X_n\}_{n=1}^{\infty}$ are i.i.d. Bernoulli random variables (i.e., X_n takes values $\{-1,1\}$ with probability 1/2). The study of such distribution has a long history (see, e.g., [E, G, S, Wi]). It follows from a theorem of Jensen and Wintner that F is either purely absolutely continuous or purely singular. It is also known that for $\rho=2^{-1/n}$, $n=1,2,\ldots$ [Wi], or for almost all ρ close enough to 1 [E], then the distribution of F is absolutely continuous, and $F'\in L^2(\mathbb{R})$. In this case the m.q.v. index of F is 1. On the other hand if $\rho=\theta^{-1}$ where θ is a *Pisot–Vijayaraghavan* (P.V.) *number* (i.e., $\theta>1$ is a root of an algebraic equation, and all its conjugate roots have modulus less than 1), then F is purely singular. A general classification of F between these two types is still open.

Our second main result is to evaluate the precise α for the self-similar measure μ with $\rho=(\sqrt{5}-1)/2$ (note that ρ^{-1} is a P.V. number, it is a root of $x^2-x-1=0$) (Theorem 4.4): For $\rho=(\sqrt{5}-1)/2$, the m.q.v. index α of μ is given by

$$(4\rho^{\alpha})^3-2(4\rho^{\alpha})^2-2(4\rho^{\alpha})+2=0 \tag{1.9}$$

($\alpha=0.9923995\ldots$). Moreover (1.8) also holds for such μ and α.

The main idea of the proof is to use the invariant property of μ to derive some identities for the α-m.q.v. (Lemma 4.6), which eventually reduces to the well known *renewal equation* $f=f*\nu+S$ on $[0,\infty)$, where ν, S are given, ν is a probability measure, and S is a "directly" Riemann integrable function [Fe]. The solution f is known and α can hence be found as in (1.9).

The formula obtained in (1.9) is markedly different from (1.7), and a general pattern for the m.q.v. index of the invariant measures for $1/2<\rho<1$ is not known.

We organize the paper as follows: in Section 2 we will define certain mean variations of μ and show that they are the necessary and sufficient conditions for (1.6) to hold. In Section 3 we use certain types of Tauberian theorems (which are proved in [LW]) to establish (1.1)′ and its variants. The results on self-similar meaures are proved in Section 4. Some further remarks and open problems in connection with the random variable $\sum_{n=1}^{\infty}\rho^n X_n$, $1/2<\rho<1$, and the Hausdorff dimension of the graph of $\sum_{n=1}^{\infty}\rho^n R_n$ (R_n's are the Rademacher functions on [0, 1]) are also discussed. Finally we give an appendix which is an interpretation of the proof of the main lemma (Lemma 4.6) for (1.9) by symbolic dynamic diagrams.

2. Mean p-Variations

We will use $|E|$ to denote the Lebesgue measure on any Borel subset in $\mathbb{R}^d$, and $Q_h(x)$ the half open cube $\prod_{j=1}^d (x_i - h, x_i + h]$, where $x = (x_1, ..., x_d)$, $h > 0$.

Lemma 2.1. *Let μ be a σ-finite Borel measure on $\mathbb{R}^d$, and let g be a Borel measurable function. Suppose $g(\cdot)\,\mu(Q_h(\cdot))$ is integrable with respect to the Lebesgue measure, then*

$$\int_{\mathbb{R}^d} g(u)\,\mu(Q_h(u))\,du = \int_{\mathbb{R}^d} \int_{Q_h(u)} g(t)\,dt\,d\mu(u).$$

Proof. It follows directly from the Fubini theorem.

Lemma 2.2. *Let μ be a positive σ-finite Borel measure on $\mathbb{R}^d$, then for any $a \in \mathbb{R}^d$, $h > 0$,*

$$\frac{1}{2^d}\,\mu(Q_h(a)) \leqslant \frac{1}{(2h)^d} \int_{Q_h(a)} \mu(Q_h(u))\,du \leqslant \mu(Q_{2h}(a)).$$

Proof. Let E_j, $j = 1, ..., 2^d$, denote the 2^d quadrants of $Q_h(a)$, then $Q_h(u) = \bigcup_{j=1}^{2^d} E_j$, and $u \in E_j$ implies that $E_j \subseteq Q_h(u)$. Hence

$$\mu(E_j) = \frac{1}{h^d} \int_{E_j} \mu(E_j)\,du \leqslant \frac{1}{h^d} \int_{E_j} \mu(Q_h(u))\,du,$$

and the first inequality follows. For the second inequality, we observe that $Q_h(u) \subseteq Q_{2h}(a)$ for any $u \in Q_h(a)$, so that

$$\frac{1}{(2h)^d} \int_{Q_h(a)} \mu(Q_h(u))\,du \leqslant \frac{1}{(2h)^d} \int_{Q_h(a)} \mu(Q_{2h}(a))\,du = \mu(Q_{2h}(a)).$$

For $0 \leqslant \alpha \leqslant d$, let $\mathfrak{M}_\alpha^p$ be the class of complex valued σ-finite Borel measures μ on $\mathbb{R}^d$ such that

$$\|\mu\|_{\mathfrak{M}_\alpha^p} := \sup_{0 < h \leqslant 1} \left(\frac{1}{(2h)^{d + \alpha(p-1)}} \int_{\mathbb{R}^d} |\mu|(Q_h(u))^p\,du \right)^{1/p} < \infty \tag{2.1}$$

if $1 \leqslant p < \infty$, and

$$\|\mu\|_{\mathfrak{M}_\alpha^\infty} := \sup_{u \in \mathbb{R}^d} \sup_{0 < h \leqslant 1} \frac{1}{(2h)^\alpha} |\mu|(Q_h(u)) < \infty$$

if $p=\infty$, where $|\mu|$ denotes the total variation of μ. Note that for $1<p\leqslant\infty$, $\mathfrak{M}_\alpha^p$ is a normed linear space but not complete. That $\mu\in\mathfrak{M}_\alpha^\infty$ is equivalent to $|\mu|$ being locally uniformly α-dimensional; for $p=1$, Lemma 2.1 implies that

$$\begin{aligned}\|\mu\|_{\mathfrak{M}_\alpha^1} &= \sup_{0<h\leqslant 1}\frac{1}{(2h)^\alpha}\int_{\mathbb{R}^d}|\mu|Q_h(u))\,du\\ &= \sup_{0<h\leqslant 1}\frac{1}{(2h)^\alpha}\int_{\mathbb{R}^d}\int_{Q_h(u)}1\,dt\,d|\mu|\,(u)\\ &= |\mu|(\mathbb{R}^d);\end{aligned}$$

for $\alpha=d$, $1<p<\infty$, it follows from [HL] that $\mu\in\mathfrak{M}_\alpha^p$ implies that μ is absolutely continuous and $d\mu/dx=g$ is in $L^p(\mathbb{R})$ and

$$\|\mu\|_{\mathfrak{M}_d^p}=\lim_{h\to 0}\frac{1}{(2h)^d}\left(\int_{\mathbb{R}^d}|\mu|(Q_h(u))^p\,du\right)^{1/p}=\|g\|_p.$$

By using Lemma 2.2, it is easy to see that for $1\leqslant p<\infty$,

$$\mu\in\mathfrak{M}_\alpha^p \quad\Leftrightarrow\quad \sup_{0<h\leqslant 1}(2h)^{-\alpha(p-1)}\sum_a|\mu|(Q_h(a))^p<\infty, \tag{2.1$'$}$$

the summation is taken over all the a's belonging to the h-mesh, i.e., $a\in(2h)\mathbb{Z}^d$. The class $\mathfrak{M}_\alpha^p$ in the form of (2.1)$'$ has been used to study the theory of multifractals (see [F2]).

For any Borel measure μ and for any Borel measure measurable function f on $\mathbb{R}^d$, we use μ_f to denote the measure such that $d\mu_f=f\,d\mu$.

THEOREM 2.3. *Let $1\leqslant p\leqslant q\leqslant\infty$, $0\leqslant\alpha\leqslant d$, and let μ be a positive σ-finite Borel measure, then $\mu_f\in\mathfrak{M}_\alpha^p$ for all $f\in L^q(\mu)$ with*

$$\|\mu_f\|_{\mathfrak{M}_\alpha^p}\leqslant C\,\|f\|_{L^q(\mu)}$$

for some $C>0$ if and only if $\mu\in\mathfrak{M}_\alpha^r$, $r=p(q-1)/(q-p)$ ($r=1$ if $p=q=1$; $r=\infty$ if $p=q=\infty$).

Proof. We will consider the case $1<p\leqslant q<\infty$ only, the cases $p=1$, or $q=\infty$ only need some obvious adjustments. For simplicity we will make use of the modulus of μ in (2.1)$'$.

To prove the sufficiency, we note that

$$|\mu|(Q_h(u))\leqslant\left(\int_{Q_h(u)}|f|^q\,d\mu\right)^{1/q}\cdot\mu(Q_h(u))^{1/q'},$$

where $1/q+1/q'=1$. The Hölder inequality hence implies that

$$h^{-\alpha(p-1)}\sum_a |\mu|(Q_h(a))^p$$

$$\leqslant h^{-\alpha(p-1)}\sum_a\left(\left(\int_{Q_h(a)}|f|^q\,d\mu\right)^{p/q}\cdot\mu(Q_h(a))^{p/q'}\right)$$

$$\leqslant h^{-\alpha(p-1)}\left(\sum_a \mu(Q_h(a))^{p/q'\cdot q/(q-p)}\right)^{(p-q)/q}\left(\sum_a\int_{Q_h(a)}|f|^q\,d\mu\right)^{p/q}$$

$$\leqslant\left(h^{-\alpha(p-1)}\sum_a \mu(Q_h(a))^r\right)^{pr}\cdot\|f\|^p_{L^q(m)}.$$

Since $\mu\in\mathfrak{M}^r_\alpha$, it follows that $\mu_f\in\mathfrak{M}^p_\alpha$ and (2.2) follows.

For the reverse inequality, we let

$$f=\sum_{a\in A}\mu(Q_h(a))^{(p-1)/(q-p)}\cdot\chi_{Q_h(a)},$$

where A is a finite subset of the $2h$-mesh, then

$$\|f\|_{L^q(\mu)}=\left(\sum_{a\in A}\mu(Q_h(a))^r\right)^{1/q},$$

and $\|\mu_f\|_{\mathfrak{M}^p_\alpha}$ is equivalent to

$$\sup_{0<h\leqslant 1}\left(h^{-\alpha(p-1)}\sum_{a'}|\mu_f|\,(Q_h(a'))^p\right)^{1/p}$$

$$=\sup_{0<h\leqslant 1}\left(h^{-\alpha(p-1)}\sum_{a'}\left\{\sum_{a\in A}\mu(Q_h(a))^{(p-1)/(q-p)}\int_{Q_h(a')}\chi_{Q_h(a)}\,d\mu\right\}\right)^{1/p}$$

$$=\sup_{0<h\leqslant 1}\left(h^{-\alpha(p-1)}\sum_{a\in A}\mu(Q_h(a))^{(p-1)/(q-p)}\mu(Q_h(a))\right)^{1/p}$$

$$=\sup_{0<h\leqslant 1}\left(h^{-\alpha(p-1)}\sum_{a\in A}\mu(Q_h(a))^r\right)^{1/p}.$$

The assumption $\|\mu_f\|_{\mathfrak{M}^p_\alpha}\leqslant C\,\|f\|_{L^q(\mu)}$ yields

$$\sup_{0<h\leqslant 1}\left(h^{-\alpha(p-1)}\sum_{a\in A}\mu(Q_h(a))^r\right)^{1/p}\leqslant C_1\left(\sum_{a\in A}\mu(Q_h(a))^r\right)^{1/q}.$$

A direct calculation hence implies that

$$\sup_{0<h\leqslant 1}h^{-\alpha(p-1)}\sum_{a\in A}\mu(Q_h(a))^r\leqslant C_1^{q/(q-1)}.$$

Since A is an arbitrary finite subset of the $2h$-mesh, we can now take the sum over all the a's in the h-mesh, and hence $\mu \in \mathfrak{M}_\alpha^r$.

As special cases of the above theorem we have

COROLLARY 2.4. *Let $0 \leqslant \alpha \leqslant d$, and let μ be a positive σ-finite Borel measure on $\mathbb{R}^d$, then*

(i) *μ is locally uniformly α-dimensional if and only if there exists $p > 1$ (and hence all $p > 1$), and $C > 0$ (depends on p) such that*

$$\|\mu_f\|_{\mathfrak{M}_\alpha^p} \leqslant C \|f\|_{L^p(\mu)} \qquad \text{for all} \quad f \in L^p(\mu).$$

(ii) *For $1 \leqslant p \leqslant \infty$, $\mu \in \mathfrak{M}_\alpha^p$ if and only if there exists $C > 0$ such that*

$$\|\mu_f\|_{\mathfrak{M}_\alpha^p} \leqslant C \|f\|_{L^\infty(\mu)} \qquad \text{for all} \quad f \in L^\infty(\mu).$$

Let μ and ν be two positive measures on $\mathbb{R}^d$, we say that μ is *null* with respect to ν ($\mu \lll \nu$) if for any Borel subset E, $\mu(E) < \infty$ implies that $\nu(E) = 0$. This definition was introduced by Strichartz [Str2], he proved that

THEOREM 2.5. *Let μ, ν be positive Borel mesures. Suppose μ is σ-finite, ν has no infinite atom, and $\mu \ll \nu$, then $\mu = \mu_1 + \mu_2$ where $d\mu_1 = \phi \, d\nu$ for some Borel measurable ϕ, and $\mu_2 \lll \nu$.*

For any positive Borel measure μ, we use $L_\sigma^1(\mu)$ to denote the class of Borel measurable functions f such that $\{x : f(x) \neq 0\} = \bigcup_{n=1}^\infty E_n$ and $f/E_n \in L^1(\mu)$. Let ω_α be the α-dimensional Hausdorff measure on $\mathbb{R}^d$. It is clear that if $\mu \in \mathfrak{M}_\alpha^\infty$, i.e., μ is locally uniformly α-dimensional, then $\mu \ll \omega_\alpha$ and hence $\mu = \phi \, d\omega_\alpha + \nu$ where $\phi \in L_\sigma^1(\omega_\alpha)$, and $\nu \lll \omega_\alpha$. We can relax the condition on μ as following:

PROPOSITION 2.6. *Let $0 \leqslant \alpha \leqslant d$. Let $\mu \geqslant 0$ be a σ-finite Borel measure on $\mathbb{R}^d$. Suppose*

$$\Phi(x) = \sup_{0 < h \leqslant 1} \frac{1}{(2h)^\alpha} \mu(Q_h(x)), \qquad x \in \mathbb{R}^d,$$

is finite for μ-almost all x, then $\mu \ll \omega_\alpha$ and μ has a decomposition $\mu = \phi \, d\omega_\alpha + \nu$ where $\phi \in L_\sigma^1(\omega_\alpha)$, and $\nu \lll \omega_\alpha$.

Proof. For any integer k, let

$$E_k = \left\{ x \in \mathbb{R}^d : 2^k < \sup_{0 < h \leqslant 1} \frac{1}{(2h)^\alpha} \mu(Q_h(x)) \leqslant 2^{k+1} \right\},$$

and let $\mu_k = \mu/E_k$, then $\mu = \sum_{k=-\infty}^{\infty} \mu_k$ and μ_k is locally uniformly α-dimensional with bound 2^{k+1}. It follows from Theorem 2.5 that $\mu_k = \phi_k \, d\omega_\alpha + \nu_k$, and $\phi_k \in L^1_\sigma(\omega_\alpha)$, $\nu_k \lll \omega_\alpha$. The proposition follows by letting $\phi = \sum \phi_k$ and $\nu = \sum \nu_k$.

Let $\bar{D}_\alpha(\mu, x) = \limsup_{h \to 0} \mu(B_h(x))/(2h)^\alpha$ denote the *α-upper density* of μ at x, and similarly, let $\underline{D}_\alpha(\mu, x) = \liminf_{h \to 0} \mu(B_h(x))/(2h)^\alpha$ denote the *α-lower density* of μ at x.

PROPOSITION 2.7. *Let μ, Φ, and ϕ be as in Proposition 2.6. For $\varepsilon > 0$, there exists a Borel subset F such that $\mu(F) < \varepsilon$, and $\bar{D}_\alpha(\mu/F^c, x) \leqslant \phi(x)$ for $x \in F^c$ (the complement of F in $\mathbb{R}^d$).*

Proof. Let $E = \{x \in \mathbb{R}^d : \phi(x) \neq 0\}$, then E is a $\omega_\alpha - \sigma$-finite set, we can write E as a disjoint union of $\{E_j\}$ with $0 < \omega_\alpha(E_j) < \infty$ and $\int_{E_j} \phi \, d\omega_\alpha < \infty$. It follows from [F1, Corollary 2.5] that

$$\bar{D}_\alpha(\omega_\alpha/E_j, x) \begin{cases} \leqslant 1 & \text{for } \omega_\alpha\text{-almost all } x \in E_j \\ = 0 & \text{otherwise.} \end{cases}$$

This and [Str2, Corollary 2.3] imply that

$$\limsup_{h \to 0} \frac{1}{(2h)^\alpha} \int_{B_h(x) \cap E_j} \phi \, d\omega_\alpha \leqslant \chi_{E_j}(x) \, \phi(x) \tag{2.2}$$

for ω_α-almost all $x \in \mathbb{R}^d$. Also note that

$$\limsup_{h \to 0} \frac{\nu(B_h(x))}{(2h)^\alpha} = 0 \qquad \text{for } \omega_\alpha\text{-almost all } x \tag{2.3}$$

[Str2, Theorem 3.2]. Since $\mu \ll \omega_\alpha$, we can replace the statements in (2.2) and (2.3) by μ-almost all x. For $\varepsilon > 0$, we can choose j_0 such that $\mu(\bigcup_{j=j_0+1}^{\infty} E_j) < \varepsilon$. Let F be the union of $\bigcup_{j=j_0+1}^{\infty} E_j$ and the μ-zero sets occurs in (2.2), (2.3). Then for $x \in F^c$, we have

$$\begin{aligned} \bar{D}_\alpha(\mu/F^c, x) &= \limsup_{h \to 0} \frac{\mu(B_h(x) \cap F^c)}{(2h)^\alpha} \\ &\leqslant \limsup_{h \to 0} \frac{1}{(2h)^\alpha} \left(\int_{B_h(x) \cap F^c} \phi \, d\omega_\alpha + \nu(B_h(x)) \right) \\ &\leqslant \sum_{j=1}^{j_0} \limsup_{h \to 0} \frac{1}{(2h)^\alpha} \int_{B_h(x) \cap E_j} \phi \, d\omega_\alpha \\ &\leqslant \phi(x). \end{aligned}$$

If we replace the $\sup_{0<h\leqslant 1}$ in Theorem 2.3 by $\limsup_{h\to 0}$, we have

THEOREM 2.8. *Let $1\leqslant p\leqslant q\leqslant\infty$, and let $0\leqslant\alpha\leqslant d$. Suppose μ is a positive σ-finite Borel measure on $\mathbb{R}^d$, then*

$$\limsup_{h\to 0}\left(\frac{1}{(2h)^{d+\alpha(p-1)}}\int_{\mathbb{R}^d}|\mu_f|(Q_h(u))^p\,du\right)^{1/p}$$
$$\leqslant C\,\|f\|_{L^q(\phi\,d\omega_\alpha)}\qquad\forall f\in L^q(\mu)\tag{2.4}$$

provided that

$$\Phi(x)=\sup_{0<h\leqslant 1}\frac{1}{(2h)^\alpha}\mu(Q_h(x))$$

is in $L^s(\mu)$ where $s=(p-1)q/(q-p)$. ($s=(p-1)$ if $q=\infty$; $L^0(\mu)$ just means the class of Borel measurable functions by convention.)

Proof. We consider the case $1<p<q<\infty$ only. Note that $s=r-1$ where r is defined in Theorem 2.3. Since

$$\frac{1}{(2h)^{d+\alpha(r-1)}}\int_{\mathbb{R}^d}\mu(Q_h(u)^r\,du$$
$$=\frac{1}{(2h)^{d+\alpha(r-1)}}\int_{\mathbb{R}^d}\int_{Q_h(u)}\mu(Q_h(t))^{r-1}\,dt\,d\mu(u)\qquad\text{(by Lemma 2.1)}$$
$$\leqslant\frac{1}{(2h)^{\alpha s}}\int_{\mathbb{R}^d}\mu(Q_{2h}(u))^s\,d\mu(u)\qquad\text{(by Lemma 2.2)}$$
$$\leqslant 2^{\alpha s}\int_{\mathbb{R}^d}(\Phi(u))^s\,d\mu(u).\tag{2.5}$$

It follows that $\Phi\in L^s(\mu)$ implies that $\mu\in\mathfrak{M}^r_\alpha$. Write $\mu=\phi\,d\omega_\alpha+\nu$ as in Proposition 2.6, then both $\phi\,d\omega_\alpha$ and ν are in $\mathfrak{M}^r_\alpha$.

Let $f\in L^q(\mu)$, then $f\in L^q(\nu)$ and by using the same argument as in the proof of the sufficiency of Theorem 2.3 and as in (2.5), we have

$$\limsup_{h\to 0}\frac{1}{(2h)^{d+\alpha(p-1)}}\int_{\mathbb{R}^d}|\nu_f|(Q_h(u))^p\,du$$
$$\leqslant C\limsup_{h\to 0}\left(\frac{1}{(2h)^{\alpha s}}\int_{\mathbb{R}^d}\nu(Q_h(u))^s\,du\right)\cdot\|f\|^p_{L^q(\nu)}$$
$$\leqslant C_1\limsup_{h\to 0}\left(\int_{\mathbb{R}^d}(\nu(Q_h(u))/(2h)^\alpha)^s\,du\right)\cdot\|f\|^p_{L^q(\nu)}.\tag{2.6}$$

Observe that $\nu(Q_h(u))/(2h)^\alpha \to 0$ as $h \to 0$ for ω_α-almost all u [Str2, Theorem 3.2], and hence for μ-almost all u (since $\mu \ll \omega_\alpha$). Since $\nu(Q_h(u))/(2h)^\alpha \leqslant \Phi(u)$ and $\Phi \in L^s(\nu)$, the dominated convergence theorem implies that the limit in (2.6) tends to 0 as $h \to 0$. We hence have by the Minkowski inequality, and (2.6) that

$$\begin{aligned}\limsup_{h \to 0} &\frac{1}{(2h)^{d+\alpha(p-1)}} \int_{\mathbb{R}^d} |\mu_f| (Q_h(u))^p \, du \\ &= \limsup_{h \to 0} \frac{1}{(2h)^{d+\alpha(p-1)}} \int_{\mathbb{R}^d} |\tilde{\mu}_f| (Q_h(u))^p \, du, \qquad (2.7)\end{aligned}$$

where $\tilde{\mu} = \phi \, d\omega_\alpha$. Again by repeating the same argument as in (2.6), the last expression is

$$\begin{aligned}&\leqslant C_1 \limsup_{h \to 0} \left(\int_{\mathbb{R}^d} (\mu_f(Q_h(u))/(2h)^\alpha)^s \, du \right) \cdot \|f\|^p_{L^q(\phi \, d\omega_\alpha)} \\ &\leqslant C_2 \int_{\mathbb{R}^d} (\Phi(u))^s \, d\mu(u) \cdot \|f\|^p_{L^q(\phi \, d\omega_\alpha)} \\ &\leqslant C_3 \|f\|^p_{L^q(\phi \, d\omega_\alpha)},\end{aligned}$$

and the theorem is proved.

We have a partial result for the reverse inequality of the above theorem. First we establish a simple lemma.

LEMMA 2.9. *Let $\mu \geqslant 0$ be a σ-finite Borel measure on $\mathbb{R}^d$, and let $f \in L^p(\mu)$, $1 \leqslant p < \infty$, then*

$$\lim_{h \to 0} \frac{1}{(2h)^d} \int_{Q_h(u)} (u_f(Q_h(t))/\mu(Q_h(u))) \, dt = f(u) \qquad \text{in } L^p(\mu).$$

Proof. On $\mathbb{R}^d \times \mathbb{R}^d$, let

$$A_h(u, v) = \{(s, t) : s \in Q_h(u - v + t), t \in Q_h(v)\}$$

be the parallelopiped centered at (u, v), let ν be the product measure of μ and the Lebesgue measure on $\mathbb{R}^d$, and let F be defined by

$$F(u, v) = \begin{cases} f(u) & \text{if } v = u + w \quad |w| < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\frac{1}{(2h)^d}\int_{Q_h(u)} (\mu_f(Q_h(t))/\mu(Q_h(u)))\,dt$$

$$=\frac{1}{(2h)^d\mu(Q_h(u))}\int_{Q_h(u)}\int_{Q_h(t)} f(s)\,d\mu(s)\,dt$$

$$=\frac{\nu(A_h(u,u))}{(2h)^d\mu(Q_h(u))}\cdot\frac{1}{\nu(A_h(u,u))}\int_{A_h(u,u)} F(s,t)\,d\nu(s,t).$$

Note that the first factor is bounded, the second factor equals

$$\frac{1}{\nu(A_h(u,v))}\int_{A_h(u,v)} F(s,t)\,d\nu(s,t),$$

for $v=u+w$ with $|w|<1$. It follows from [Str2, Corollary 2.4] that the above expression converges to $F(u,v)$ in $L^p(\nu)$ (we are using the parallelopipeds instead of the balls). Hence

$$\lim_{h\to 0}\frac{1}{(2h)^d}\int_{Q_h(u)} (\mu_f(Q_h(t))/\mu(Q_h(u)))\,dt=f(u) \qquad \text{in } L^p(\mu).$$

THEOREM 2.10. *Let* $0\leqslant\alpha\leqslant d$. *Suppose* $\mu\geqslant 0$ *and* $\mu\in\mathfrak{M}_\alpha^\infty$ *with* $\underline{D}_\alpha(\mu,x)\geqslant C>0$ *for* μ*-almost all* x, *then*

$$\liminf_{h\to 0}\left(\frac{1}{(2h)^{d+\alpha}}\int_{\mathbb{R}^d}|\mu_f|(Q_h(u))^2\,du\right)^{1/2}\geqslant C\,\|f\|_{L^2(\phi\,d\omega_\alpha)} \qquad \forall f\in L^2(\mu).$$

In particular if $\underline{D}_\alpha(\mu,x)<\bar{D}_\alpha(\mu,x)=C$ *for* μ*-almost all* x, *then*

$$\lim_{h\to 0}\left(\frac{1}{(2h)^{d+\alpha}}\int_{\mathbb{R}^d}|\mu_f|(Q_h(u))^2\,du\right)^{1/2}=C\,\|f\|_{L^2(\phi\,d\omega_\alpha)} \qquad \forall f\in L^2(\mu).$$

Proof. Let $\tilde{\mu}=\phi\,d\omega_\alpha$ be as in Theorem 2.8, then (2.7) remains valid by replacing lim sup with lim inf, i.e.,

$$\liminf_{h\to 0}\frac{1}{(2h)^{d+\alpha}}\int_{\mathbb{R}^d}|\mu_f|(Q_h(u))^2\,du$$

$$=\liminf_{h\to 0}\frac{1}{(2h)^{d+\alpha}}\int_{\mathbb{R}^d}|\tilde{\mu}_f|(Q_h(u))^2\,du.$$

By Lemma 2.1, we can express the last integral as

$$\int_{\mathbb{R}^d} |\tilde{\mu}_f| (Q_h(u))^2 \, du = \int_{\mathbb{R}^d} \left(\int_{Q_h(u)} \tilde{\mu}_f (Q_h(t)) \, dt \right) \overline{f(u)} \, \phi(u) \, du$$

$$= (2h)^d \int_{\mathbb{R}^d} (f(u) + \varepsilon_h(u)) \, \tilde{\mu}(Q_h(u)) \, \overline{f(u)} \, \phi(u) \, du,$$

where

$$\varepsilon_h(u) = \frac{1}{(2h)^d} \int_{Q_h(u)} (f(u) - [\tilde{\mu}_f (Q_h(t)) / \tilde{\mu}_f (Q_h(u))]) \, dt.$$

Hence by Lemma 2.9,

$$\lim_{h \to 0} \left| \frac{1}{(2h)^{d+\alpha}} \int_{\mathbb{R}^d} (2h)^d \varepsilon_h(u) \, \tilde{\mu}(Q_h(u)) \, \overline{f(u)} \, \phi(u) \, d\omega_\alpha(u) \right|$$

$$\leqslant \lim_{h \to 0} \left(\int_{\mathbb{R}^d} |\varepsilon_h(u)|^2 \, \phi(u) \, d\omega_\alpha(u) \right)^{1/2} \cdot \|\mu\|_{\mathfrak{M}^2_\alpha} \cdot \|f\|_{L^2(\phi \, d\omega_\alpha)} = 0.$$

We concluded that

$$\liminf_{h \to 0} \frac{1}{(2h)^{d+\alpha}} \int_{\mathbb{R}^d} |\mu_f (Q_h(u))|^2 \, du$$

$$= \liminf_{h \to 0} \int_{\mathbb{R}^d} \frac{\tilde{\mu}(Q_h(u))}{(2h)^\alpha} \, |f(u)|^2 \, \phi(u) \, d\omega_\alpha(u),$$

and the assertions follows.

3. The Fourier Transformation

For $1 \leqslant p < \infty$, $0 \leqslant \alpha < d$, we let

$$\mathfrak{B}^p_\alpha = \left\{ f \in L^p_{\text{loc}}(\mathbb{R}^d) : \|f\| = \sup_{T \geqslant 1} \left(\frac{1}{T^{d-\alpha}} \int_{B_T} |f|^p \right)^{1/p} < \infty \right\},$$

then $\mathfrak{B}^p_\alpha$ is a Banach space, and for $0 \leqslant \alpha \leqslant \beta < n$,

$$\mathfrak{B}^p_\alpha \subseteq \mathfrak{B}^p_\beta \subseteq \mathfrak{B}^p_0 \subseteq L^p(dx/(1 + |x|^{n+1}))$$

[LW, Proposition 4.2]. For $h > 0$, we define the transformation W_h as

$$(W_h f)(x) = \int_{\mathbb{R}^d} f(y) \, E_h(y) \, e^{2\pi i x \cdot y} \, dy,$$

where

$$E_h(y) := \int_{|\xi| \leqslant h} e^{2\pi i y \xi} \, d\xi = 2\pi \left(\frac{h}{|y|}\right)^{d/2} J_{d/2}(2\pi h \, |y|),$$

and $J_{d/2}$ is the Bessel function of order $d/2$. The main purpose for defining such transformation is that if μ is a bounded Borel measure on $\mathbb{R}^d$, and $f = \hat{\mu}$, then for $h > 0$, and for any ball $B_h(x)$, $\mu(B_h(\cdot))^\wedge = (\mu * \chi_{B_h})^\wedge = \hat{\mu} \cdot E_h$. It follows that

$$\mu(B_h(x)) = (W_h f)(x) \tag{3.1}$$

for Lebesgue-almost all $x \in \mathbb{R}^d$. The following theorem is proved in [LW, Theorem 4.4, Corollary 4.11]:

THEOREM 3.1. *Let $f \in \mathfrak{B}^2_\alpha$, we have*

$$\text{(i)} \qquad \sup_{T \geqslant 1} \frac{1}{T^{d-\alpha}} \int_{B_T} |f|^2 \approx \sup_{0 < h \leqslant 1} \frac{1}{h^{d+\alpha}} \int_{\mathbb{R}^d} |W_h f|^2$$

$$\text{(ii)} \quad \limsup_{T \to \infty} \frac{1}{T^{d-\alpha}} \int_{B_T} |f|^2 \approx \limsup_{h \to 0} \frac{1}{h^{d+\alpha}} \int_{\mathbb{R}^d} |W_h f|^2$$

$$\text{(iii)} \qquad \lim_{T \to \infty} \frac{1}{T^{d-\alpha}} \int_{B_T} |f|^2 = C_\alpha \lim_{h \to 0} \frac{1}{h^{d+\alpha}} \int_{\mathbb{R}^d} |W_h f|^2$$

for some $C_\alpha > 0$ independent of f, provided that either one of the limits exists.

Part (iii) of the above theorem can be extended to the following case involving the periodic functions [LW, Theorem 4.10] which will be used in Theorem 4.4.

THEOREM 3.2. *For $f \in \mathfrak{B}^2_\alpha$, the following two statements are equivalent:*

(i) *there exists a bounded multiplicative periodic function p of period $\lambda > 0$ (i.e., $p(s) = p(\lambda s)$, $s > 0$) such that*

$$\lim_{h \to 0} \left(\frac{1}{h^{d+\alpha}} \int_{\mathbb{R}^d} |W_h f|^2 - p(h) \right) = 0;$$

(ii) *there exists a bounded multiplicative periodic function q of period $\lambda > 0$ such that*

$$\lim_{T \to \infty} \left(\frac{1}{T^{d-\alpha}} \int_{B_T} |f|^2 - q(T) \right) = 0.$$

Theorem 3.2(i) and (ii) can be extended to the case $1 \leqslant p \leqslant 2$:

LEMMA 3.3. *Let $1 \leqslant p \leqslant 2$, $1/p + 1/p' = 1$, and $0 \leqslant \alpha \leqslant d$, then we have*

$$\sup_{T \geqslant 1} \left(\frac{1}{T^{d-\alpha}} \int_{B_T} |f|^{p'} \right)^{1/p'} \leqslant C \sup_{0 < h \leqslant 1} \left(\frac{1}{h^{d+\alpha(p-1)}} \int_{\mathbb{R}^d} |W_h f|^p \right)^{1/p} \tag{3.2}$$

and

$$\limsup_{T \to \infty} \left(\frac{1}{T^{d-\alpha}} \int_{B_T} |f|^{p'} \right)^{1/p'} \leqslant C \limsup_{h \to 0} \left(\frac{1}{h^{d+\alpha(p-1)}} \int_{\mathbb{R}^d} |W_h f|^p \right)^{1/p} \tag{3.3}$$

for some $C > 0$ independent of f.

Proof. It follows from the definition of W_h and the Hausdorff–Young inequality that for $1 < p \leqslant 2$,

$$\begin{aligned}
&\left(\int_{\mathbb{R}^d} |W_h f|^p \right)^{1/p} \\
&\quad \geqslant \left(\int_{\mathbb{R}^d} |f(y) E_h(y)|^{p'} dy \right)^{1/p'} \\
&\quad = \left(\int_0^\infty \left(\int_{S_{d-1}} |f(ry)|^{p'} dy \right) \left(2\pi \left(\frac{h}{r} \right)^{d/2} J_{d/2}(2\pi hr) \right)^{p'} r^{d-1} \, dr \right)^{1/p'} \\
&\quad = h^{d/p + \alpha/p'} \left(\int_0^\infty F(r/h) \, r^{d-\alpha-1} w(r) \, dr \right)^{1/p'},
\end{aligned}$$

where $S_{d-1} = \{ y \in \mathbb{R}^d : |y| = 1 \}$, $F(r) = r^\alpha \int_{S_{d-1}} |f(ry)|^{p'} dy$, and $w(r) = (2\pi r^{-d/2} J_{d/2}(2\pi r))^{p'}$. Hence

$$\left(\frac{1}{h^{d+\alpha(p-1)}} \int_{\mathbb{R}^d} |W_h f|^p \right)^{1/p} \geqslant \left(\int_0^\infty F(r/h) \, r^{d-\alpha-1} w(r) \, dr \right)^{1/p'}.$$

On the other hand,

$$\left(\frac{1}{T^{d-\alpha}} \int_{B_T} |f|^{p'} \right)^{1/p'} = \left(\int_0^\infty F(Tr) \, r^{d-\alpha-1} \chi_{[0,1]}(r) \, dr \right)^{1/p'},$$

It follows from the identity

$$J_k(x) = \frac{2(x/2)^k}{\Gamma(k+1/2)\,\Gamma(1/2)} \int_0^1 (1-t^2)^{k-(1/2)} \cos(tx) \, dx > 0$$

for $k > 0$, $x \in [0, 1]$ that $w \geqslant C \chi_{[0,1]}$ for some $C > 0$. This implies (3.2) and (3.3).

THEOREM 3.4. *Let* $1 \leqslant p \leqslant 2$, $1/p + 1/p' = 1$, *and* $0 \leqslant \alpha < d$. *Suppose* $\mu \in \mathfrak{M}_\alpha^p$, *then* $\hat{\mu} \in \mathfrak{B}_\alpha^{p'}$ *with*

$$\|\hat{\mu}\|_{\mathfrak{B}_\alpha^{p'}} \leqslant C \|\mu\|_{\mathfrak{M}_\alpha^p},$$

and

$$\limsup_{T \to \infty} \left(\frac{1}{T^{d-\alpha}} \int_{B_T} |\hat{\mu}|^{p'} \right)^{1/p'} \leqslant C \limsup_{h \to 0} \left(\frac{1}{h^{d+\alpha(p-1)}} \int_{\mathbb{R}^d} |\mu(B_h(x))|^p \right)^{1/p}$$

for some $C > 0$ *independent of* μ.

Proof. In view of Lemma 3.3 we need only show that $\hat{\mu}$ is actually well defined as a locally p'-integrable function, and is in $\mathfrak{B}_\alpha^{p'}$. Since μ is σ-finite, there exists an increasing sequence of Borel sets $\{E_k\}$ with $\bigcup_k E_k = \mathbb{R}^d$, $|\mu|(E_k) < \infty$, and $\lim_{k \to \infty} |\mu|(\mathbb{R}^d \backslash E_k) = 0$. Let $\mu_k = \mu/E_k$, then $\{\hat{\mu}_k\}$ is a sequence of bounded continuous functions. It follows from Theorem 2.3 (taking $p = q$, $f = \chi_{\mathbb{R}^d \backslash E_k}$) that

$$\lim_{k \to \infty} \|\mu_k - \mu\|_{\mathfrak{M}_\alpha^p} \leqslant C' \lim_{k \to \infty} |\mu|(\mathbb{R}^d \backslash E_k) = 0.$$

By (3.1), Theorem 3.1(i), and Lemma 3.3, $\{\hat{\mu}_k\}$ is a Cauchy sequence in $\mathfrak{B}_\alpha^{p'}$, and hence converges to some $\psi \in \mathfrak{B}_\alpha^{p'}$. Since $\mathfrak{B}_\alpha^{p'} \subseteq L^{p'}(dx/1 + |x|^{n+1})$ with

$$\|\psi\|_{\mathfrak{B}_\alpha^{p'}} \leqslant C \|\psi\|_{L^{p'}(dx/(1+|x|^{n+1}))}$$

[LW, Proposition 4.2], $\{\hat{\mu}_k\} \to \psi$ in $L^{p'}(dx/1 + |x|^{n+1})$ also. Now let ϕ be any C^∞-function with compact support, then

$$\begin{aligned} \int_{\mathbb{R}^d} \phi \, d\mu &= \lim_{n \to \infty} \int_{\mathbb{R}^d} \phi \, d\mu_k \\ &= \lim_{n \to \infty} \int_{\mathbb{R}^d} \hat{\phi}(x) \, \hat{\mu}_k(x) \, dx = \int_{\mathbb{R}^d} \hat{\phi}(x) \, \psi(x) \, dx. \end{aligned}$$

This implies that $\hat{\mu} = \psi$, and $\hat{\mu}$ is in $\mathfrak{B}_\alpha^{p'}$.

We can now state the corresponding results of the last section in terms of Fourier asymptotics.

THEOREM 3.5. *Let* $1 \leqslant p \leqslant 2$, $p \leqslant q \leqslant \infty$, $0 \leqslant \alpha < d$, *and let* μ *be a positive* σ-*finite Borel measure on* $\mathbb{R}^d$, *then* $\mu \in \mathfrak{M}_\alpha^r$, $r = p(q-1)/(q-p)$ *implies that*

$$\|(\mu_f)^\wedge\|_{\mathfrak{B}_\alpha^{p'}} \leqslant C \|f\|_{L^q(\mu)} \qquad \forall f \in L^q(\mu) \tag{3.4}$$

for some $C > 0$. *The converse of the statement also holds for* $p = 2$.

Proof. Inequality (3.4) is a direct consequence of Theorem 2.3 and Theorem 3.4. The case for $p=2$ follows from Theorem 2.3 and Theorem 3.1(i).

COROLLARY 3.6. *Let* $1<p\leqslant 2$, $0\leqslant\alpha<d$. *Suppose* $\mu\geqslant 0$ *is locally uniformly* α*-dimensional, then there exists* $C>0$ *such that*

$$\|(\mu_f)^\wedge\|_{\mathfrak{B}_\alpha^{p'}}\leqslant C\,\|f\|_{L^p(\mu)}\qquad \forall f\in L^p(\mu).$$

The converse of the statement also holds for $p=2$.

COROLLARY 3.7. *Let* $1<p\leqslant 2$, $0\leqslant\alpha<d$, *and let* $\mu\geqslant 0$ *be a* σ*-finite Borel measure, then* $\mu\in\mathfrak{M}_\alpha^p$ *implies that there exists* $C>0$ *such that*

$$\|(\mu_f)^\wedge\|_{\mathfrak{B}_\alpha^{p'}}\leqslant C\,\|f\|_{L^\infty(\mu)}\qquad \forall f\in L^\infty(\mu).$$

THEOREM 3.8. *Let* $1\leqslant p\leqslant 2$, $p\leqslant q<\infty$, $0\leqslant\alpha<d$. *Suppose* $\mu\geqslant 0$, *and*

$$\Phi(x)=\sup_{0<h\leqslant 1}\frac{1}{(2h)^\alpha}\,\mu(Q_h(x))$$

is in $L^s(\mu)$ where $s=(p-1)q/(q-p)$, then

$$\limsup_{T\to\infty}\left(\frac{1}{T^{d-\alpha}}\int_{B_T}(|(\mu_f)^\wedge|^{p'})\right)^{1/p'}$$
$$\leqslant C\,\|f\|_{L^q(\phi\,d\omega_\alpha)}\qquad \forall f\in L^q(\mu),$$

where $\mu=\phi\,d\omega_\alpha+\nu$ as in Theorem 2.8.

COROLLARY 3.9. *Suppose* $\mu\geqslant 0$ *is locally uniformly* α*-dimensional, then*

$$\limsup_{T\to\infty}\left(\frac{1}{T^{d-\alpha}}\int_{B_T}(|(\mu_f)^\wedge|^2)\right)^{1/2}$$
$$\leqslant C\,\|f\|_{L^2(\phi\,d\omega_\alpha)}\qquad \forall f\in L^2(\mu).$$

For the limit case, we have the same result as [Str2, Theorem 5.5], which is a consequence of Theorem 2.10 and Theorem 3.1(iii).

THEOREM 3.10. *Suppose* $\mu\geqslant 0$ *is locally uniformly* α*-dimensional, and suppose* $\underline{D}_\alpha(\mu,x)=\bar{D}_\alpha(\mu,x)=C$ *for* μ*-almost all* x, *then*

$$\lim_{T\to\infty}\left(\frac{1}{T^{d-\alpha}}\int_{B_T}(|(\mu_f)^\wedge|^2)\right)^{1/2}$$
$$\leqslant C'\,\|f\|_{L^2(\phi\,d\omega_\alpha)}\qquad \forall f\in L^2(\mu).$$

In [Str2, Theorem 5.5] the $\liminf_{T\to\infty}$ case for the Fourier asymptotics corresponding to Theorem 2.10 is proved. We are not able to obtain such a result yet since we have not proved the corresponding type of statements of $\inf_{T\geqslant 1}$ and $\liminf_{T\to\infty}$ as in Theorem 3.1.

4. Self-Similar Measures

We will use $\tilde{W}(\mathbb{R})$ to denote the class of locally Riemann integrable functions f on $\mathbb{R}$ such that $\sum_{n=-\infty}^{\infty} \|f\chi_{[n,n+1]}\|_\infty < \infty$. This class of functions was introduced by Wiener to extend the Tauberian theorem on $L^1(\mathbb{R})$ ([W2], see also [T, p. 337], [LW]). It is also important in the renewal theory, as is given by the following elegant theorem ([Fe, p. 348], where $f \in \tilde{W}(\mathbb{R})$ is called a *directly* integrable function).

Theorem 4.1. *Let $\sigma \neq \delta_0$ be a probability measure on $[0, \infty)$, and let S be a bounded Borel measurable function on $[0, \infty)$. Suppose f is Borel measurable, bounded on $[0, s)$ for all $s > 0$, and satisfies the* renewal equation

$$f(x) = f * \sigma(x) + S(x) \qquad \left(= \int_0^x f(x-y)\, d\sigma(y) + S(x)\right),$$

on $[0, \infty)$, then $f = \sum_{n=-\infty}^{\infty} S * \sigma^n$. If in addition $S \in \tilde{W}(\mathbb{R})$, then

(i) *if σ is non-arithmetic, then $f(x) = c + o(1)$ as $x \to \infty$ where $c = (\int_0^\infty y\, d\sigma)^{-1} \cdot \int_0^\infty S(y)\, dy$;*

(ii) *if σ is arithmetic, let $a\mathbb{Z}$, $a > 0$, be the lattice generated by the support of σ, then $f(x) = p(x) + o(1)$ where $p(x) = a(\int_0^\infty y\, d\sigma)^{-1} \cdot \sum_{k=0}^{\infty} S(x + ka)$ is a periodic function of period a.*

Let $S_i\colon \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, be defined by

$$S_1(x) = \rho_1 x, \qquad S_2(x) = \rho_2 x + (1 - \rho_2),\ x \in \mathbb{R},$$

$0 < \rho_1, \rho_2 < 1$. For $a_1 + a_2 = 1$, $a_1, a_2 > 0$, there exists a unique probability measure μ which satisfies

$$\mu = a_1 \mu \circ S_1^{-1} + a_2 \mu \circ S_2^{-1} \tag{4.1}$$

[F1]. Obviously $\operatorname{supp} \mu \subseteq [0, 1]$.

Theorem 4.2. *Let ρ_i, a_i, $i = 1, 2$, be as above with $0 < \rho_1 + \rho_2 < 1$, then the m.q.v. index α is given by*

$$\rho_1^{-\alpha} a_1^2 + \rho_2^{-\alpha} a_2^2 = 1.$$

Furthermore we have:

(i) *if* $\{-\ln \rho_1, -\ln \rho_2\}$ *is non-arithmetic, then there exists* $C>0$ *such that*

$$\lim_{h \to 0} \left(\frac{1}{h^{1+\alpha}} \int_0^1 |\mu(Q_h(x))|^2 - C \right) = 0;$$

(ii) *otherwise, let* $(\ln \lambda)\mathbb{Z}$, $\lambda > 1$, *be the lattice generated by* $\{-\ln \rho_1, -\ln \rho_2\}$, *then*

$$\lim_{h \to 0} \left(\frac{1}{h^{1+\alpha}} \int_0^1 |\mu(Q_h(x))|^2 - p(h) \right) = 0$$

for some non-zero continuous function p *such that* $p(\lambda h) = p(h)$, $h > 0$.

We remark that Theorem 4.2 in the form $(1/T^{1-\alpha}) \int_{-T}^{T} |\hat{\mu}|^2$ instead of the m.q.v. above has already been obtained in [Str4] on $\mathbb{R}^d$ with S_i, $i = 1, ..., m$, satisfying the "strong open set condition" (see also [LW] for improvements). Our approach here is quite different. The simple proof of Theorem 4.1 in the following also gives a transparent motivation for the proof of Theorem 4.4 where S_i, $i = 1, 2$, do not satisfy the open set condition.

Proof. Note that for any Borel subset E,

$$\begin{aligned} E \subseteq [0, \rho_1] &\Rightarrow \mu(E) = a_1 \mu(\rho_1^{-1} E) \\ E \subseteq [\rho_1, 1-\rho_2] &\Rightarrow \mu(E) = 0 \\ E \subseteq [1-\rho_2, 1] &\Rightarrow \mu(E) = a_2 \mu(\rho_2^{-1}(E - (1-\rho_2))). \end{aligned} \tag{4.2}$$

For $h > 0$, we define

$$\Phi(h) = \int_{-\infty}^{\infty} |\mu(Q_h(x))|^2 \, dx \qquad \text{and} \qquad \Psi(h) = \frac{1}{h^{1+\alpha}} \Phi(h).$$

Let $0 < \rho < \min\{\rho_1, \rho_2, (\rho_1 + \rho_2)/2\}$, then Ψ is bounded for $h \geq \rho$. By using (4.2), we have for $0 < h \leq \rho$,

$$\begin{aligned} \Phi(h) &= \int_{-\infty}^{\rho_1 + h} |\mu(Q_h(x))|^2 + \int_{(1-\rho_2)-h}^{\infty} |\mu(Q_h(x))|^2 \\ &= a_1^2 \int_{-\infty}^{\rho_1 + h} |\mu(Q_{h/\rho_1}(\rho_1^{-1} x))|^2 + a_2^2 \int_{(1-\rho_2)-h}^{\infty} |\mu(Q_{h/\rho_2}(\rho_2^{-1} x))|^2 \\ &= \rho_1 a_1^2 \int_{-\infty}^{1 + h/\rho_1} |\mu(Q_{h/\rho_1}(x))|^2 + \rho_2 a_2^2 \int_{-h/\rho_2}^{\infty} |\mu(Q_{h/\rho_2}(x))|^2 \\ &= \rho_1 a_1^2 \Phi(h/\rho_1) + \rho_2 a_2^2 \Phi(h/\rho_2). \end{aligned}$$

Hence

$$\Psi(h)=\rho_1^{-\alpha}a_1^2\Psi(h/\rho_1)+\rho_2^{-\alpha}a_2^2\Psi(h/\rho_2), \qquad 0<h\leqslant\rho.$$

By letting $f(x)=\Psi(e^{-x+\ln\rho})$, $x=-\ln h$, we have

$$f(x)=\rho_1^{-\alpha}a_1^2 f(x+\ln\rho_1)+\rho_2^{-\alpha}a_2^2 f(x+\ln\rho_2), \qquad x>0,$$

so that we can rewrite, for $x\geqslant 0$,

$$f(x)=\int_{-\infty}^{0} f(x+y)\,d\sigma(y)=\int_0^x f(x-y)\,d\tilde{\sigma}(y)+S(x),$$

where σ is the measure supported by the two points $\ln\rho_1, \ln\rho_2$ with weights $\rho_1^{-\alpha}a_1^2$, $\rho_2^{-\alpha}a_2^2$, respectively, $\tilde{\sigma}(E)=\sigma(-E)$ and $S(x)=\int_{-\infty}^{-x} f(x-y)\,d\sigma(y)$. Note that f is continuous and is bounded and non-zero on $(-\infty, 0)$ (since Ψ is bounded for $h>\rho$), and σ has compact support, $S\not\equiv 0$ is hence continuous and has compact support, so that $S\in\tilde{W}(\mathbb{R})$; also note that $\int_0^\infty y\,d\sigma(y)<\infty$. If $\{-\ln\rho_1, -\ln\rho_2\}$ is non-arithmetic, then Theorem 4.1(i) applies. If $\{-\ln\rho_1, -\ln\rho_2\}$ is arithmetic and generates a lattice $(\ln\lambda)$, $\lambda>1$, then Theorem 4.1(ii) applies.

The case where $\rho_1+\rho_2>1$ is more complicated, we will take $\rho_1=\rho_2=\rho$ and $a_1=a_2=1/2$. It is useful to identify the self-similar measure in (4.1) with the distribution of the well-known Bernoulli convolution (up to a scaling and a homothetic translation) as follows.

THEOREM 4.3. *Let $\{X_n\}$ be a sequence of i.i.d. random variales where X_1 takes values $\{-1, 1\}$ with probability 1/2. Let $0<\rho<1$, then the measure induced by the random variable $X=\sum_{n=1}^\infty \rho^n X_n$ is the self-similar measure defined in (4.1) by the map*

$$S_1(x)=\rho x+\rho, \qquad S_2(x)=\rho x-\rho$$

with weights $a_1=a_2=1/2$.

Proof. We need only show that μ satisfies

$$\mu(E)=\tfrac{1}{2}\mu(S_1^{-1}(E))+\tfrac{1}{2}\mu(S_2^{-1}(E))$$

for all Borel subsets in $\mathbb{R}$, or equivalently

$$F(y)=\frac{1}{2}F\left(\frac{y}{\rho}-1\right)+\frac{1}{2}F\left(\frac{y}{\rho}+1\right), \qquad y\in\mathbb{R}, \tag{4.3}$$

where F is the distribution function of X. We can identify X_n as the Rademacher functions R_n on $[0, 1]$, hence

$$\begin{aligned} F\left(\frac{y}{\rho}-1\right) &= \left|\left\{x \in [0,1]: \sum_{n=1}^{\infty} \rho^n R_n(x) \leqslant \frac{y}{\rho}-1\right\}\right| \\ &= \left|\left\{x \in [0,1]: \sum_{n=1}^{\infty} \rho^{n+1} R_{n+1}\left(\frac{x}{2}\right) + \rho \leqslant y\right\}\right| \\ &= \left|\left\{2x : x \in \left[0, \frac{1}{2}\right], \sum_{n=2}^{\infty} \rho^n R_n(x) + \rho \leqslant y\right\}\right| \\ &= 2\left|\left\{x \in \left[0, \frac{1}{2}\right]: \sum_{n=1}^{\infty} \rho^n R_n(x) \leqslant y\right\}\right|. \end{aligned}$$

Similarly by replacing $R_n(x) = R_{n+1}((x+1)/2)$, we can show that

$$F\left(\frac{y}{\rho}+1\right) = 2\left|\left\{x \in \left(\frac{1}{2}, 1\right]: \sum_{n=1}^{\infty} \rho^n R_n(x) \leqslant y\right\}\right|,$$

and (4.3) follows.

It is clear that if $0 < \rho < 1/2$, then Theorem 4.4 implies that the m.q.v. index of the distribution of F is $\alpha = |\ln \rho/\ln 2|$. For $1/2 \leqslant \rho < 1$, F is absolutely continuous and $F' \in L^2(\mathbb{R})$ if and only if $\alpha = 1$, by a theorem of Hardy and Littlewood [HL]. On the other hand Erdös [E] and Salem [S] proved if ρ^{-1} is a Pisot–Vijayaraghavan (P.V.) number, then F is a singular distribution, and $|\hat{F}(t)| \to 0$ as $t \to \infty$. In this case,

$$\limsup_{h \to 0} \frac{1}{h^2} \int_{-\infty}^{\infty} |F(x+h) - F(x-h)|^2 \, dx = \infty.$$

In the following we will give the exact m.q.v. index α for the distribution F corresponding to $\rho = (\sqrt{5}-1)/2$. The corresponding $\rho^{-1} = (\sqrt{5}+1)/2$ is the simplest P.V. number. It satisfies the algebraic equation $\rho^2 + \rho - 1 = 0$, so that $\rho^2 = 1 - \rho$ and $\rho = (1-\rho)/\rho$. Also note that $\rho^2 < 1/2 < \rho$, and ρ^2, ρ are symmetric about 1/2, i.e., $\rho - (1/2) = (1/2) - \rho^2$.

Now let $S_i : \mathbb{R} \to \mathbb{R}$ and μ be defined as in (4.1), with $\rho_1 = \rho_2 = \rho$ and $a_1 = a_2 = 1/2$. Then $S_1^{-1}(y) = \rho^{-1} y$, $S_2^{-1}(y) = \rho^{-1} y - \rho$ so that

$$\mu(E) = \tfrac{1}{2}\mu(\rho^{-1}E) + \tfrac{1}{2}\mu(\rho^{-1}E - \rho). \tag{4.4}$$

From this we have for $E \subseteq [0, \rho^2]$, $\mu(E) = (1/2)\,\mu(\rho^{-1}E)$. It follows that

$$\mu(E) = \frac{1}{2}\mu(\rho^{-1}E) = \cdots = \frac{1}{2^{n-1}}\mu(\rho^{-(n-1)}E) \qquad \text{if} \quad E \subseteq [0, \rho^n],\ n \geqslant 2. \tag{4.4$'$}$$

Also the symmetry of μ about 1/2 implies that for any Borel subset E in $[0, 1]$,

$$\mu(E) = \mu(1 - E). \tag{4.4)''}$$

THEOREM 4.4. *Let $\rho = (\sqrt{5} - 1)/2$, and let μ be the corresponding self-similar measure. Suppose $0 < \alpha < 1$ satisfies*

$$(4\rho^\alpha)^3 - 2(4\rho^\alpha)^2 - 2(4\rho^\alpha) + 2 = 0, \tag{4.5}$$

then α $(=0.9923994...)$ is the m.q.v. index of μ. Furthermore there exist continuous multiplicative periodic functions p, $q \not\equiv 0$ with period ρ such that

(i) $$\lim_{h \to 0} \left(\frac{1}{h^{1+\alpha}} \int_0^1 |\mu(Q_h(x))|^2 - p(h) \right) = 0;$$

(ii) $$\lim_{T \to \infty} \left(\frac{1}{T^{1-\alpha}} \int_{B_T} (|\hat{\mu}|^2) - q(T) \right) = 0.$$

The proof of (i) depends on the following two technical lemmas and the renewal equation; the first lemma refers to some error estimations arising from the main identities in the second lemma. Part (ii) is a direct consequence of (i) and Theorem 3.2.

LEMMA 4.5. *Let ρ and μ be as in Theorem 4.4, then the following integrals*

$$\int_{\rho^2 - h}^{\rho^2 + h} |\mu(Q_h(x))|^2, \qquad \int_{\rho - h}^{\rho + h} |\mu(Q_h(x))|^2, \qquad \int_0^{h^{2/3}} |\mu(Q_h(x))|^2$$

are of order $o(h^\eta)$ as $h \to 0$ for some $\eta > 2$.

Proof. For $h > 0$ small enough, let N be the largest integer such that $h/\rho^{3N} < \rho^2$. Let $A(h) = \int_{\rho^2 - h}^{\rho^2 + h} |\mu(Q_h(x))|^2$, then by (4.4),

$$A(h)^{1/2} \leqslant \frac{1}{2} \left(\int_{\rho^2 - h}^{\rho^2 + h} \left| \mu\left(Q_{h/\rho}\left(\frac{x}{\rho}\right)\right)\right|^2 \right)^{1/2} + \frac{1}{2} \left(\int_{\rho^2 - h}^{\rho^2 + h} \left| \mu\left(Q_{h/\rho}\left(\frac{x}{\rho} - \rho\right)\right)\right|^2 \right)^{1/2}$$

$$= \frac{1}{2} (A_1(h)^{1/2} + A_2(h)^{1/2}), \qquad \text{say.}$$

By a change of variable $y = x/\rho - \rho$, we have

$$A_2(h) = \rho \int_{-h/\rho}^{h/\rho} |\mu(Q_{h/\rho}(y))|^2 \leqslant 2\rho \int_0^{h/\rho} |\mu(Q_{h/\rho}(y))|^2.$$

Let $E=[0, h/\rho]$, then $h/\rho^{3N}<\rho^2$ implies that $E\subseteq[0, \rho^{2N+1}]$. By (4.4)′ and a change of variable again, the last expression

$$=2\rho(1/4)^{2N}\int_0^{h/\rho}|\mu(Q_{h/\rho^{2N+1}}(y/\rho^{2N}))|^2$$

$$=2\rho(\rho/4)^{2N}\int_0^{h/\rho^{2N+1}}|\mu(Q_{h/\rho^{2N+1}}(y))|^2\leqslant(\rho/4)^{2N}.$$

Also

$$A_1(h)^{1/2}=\left(\rho\int_{\rho-h/\rho}^{\rho+h/\rho}|\mu(Q_{h/\rho}(x))|^2\right)^{1/2}$$

$$=\left(\rho\int_{\rho^2-h/\rho}^{\rho^2+h/\rho}|\mu(Q_{h/\rho}(x))|^2\right)^{1/2}\qquad\text{(by (4.4)″)}$$

$$\leqslant\frac{1}{2}\left(\rho\int_{\rho^2-h/\rho}^{\rho^2+h/\rho}\left|\mu\left(Q_{h/\rho^2}\left(\frac{x}{\rho}\right)\right)\right|^2\right)^{1/2}$$

$$+\frac{1}{2}\left(\rho\int_{\rho^2-h/\rho}^{\rho^2+h/\rho}\left|\mu\left(Q_{h/\rho^2}\left(\frac{x}{\rho}-\rho\right)\right)\right|^2\right)^{1/2}\qquad\text{(by (4.4))}$$

$$\leqslant\frac{\rho}{2}\left(\int_{\rho-h/\rho^2}^{\rho+h/\rho^2}|\mu(Q_{h/\rho^2}(x))|^2\right)^{1/2}+\frac{1}{2}(A_2(h/\rho))^{1/2}$$

$$=\frac{\rho}{2}\left(\int_{\rho^2-h/\rho^2}^{\rho^2+h/\rho^2}|\mu(Q_{h/\rho^2}(x))|^2\right)^{1/2}+\frac{1}{2}(A_2(h/\rho))^{1/2}\qquad\text{(by (4.4)″)}.$$

$$\leqslant\frac{1}{2}(A_1(h/\rho^2))^{1/2}+(\rho/4)^{N-1}.$$

A simple inductive argument implies that $A(h)=O(N^2(\rho/4)^{2N})$. Since N is the largest integer so that $h/\rho^{3N}<\rho^2$, we have $h/\rho^{3(N+1)}>\rho^2$. This implies that $h^\delta>(\rho/4)^{2N}$ where $\delta=2(\ln\rho-\ln 4)/3\ln\rho$ $(=2.587...)$. If we let $2<\eta<\delta$, then $A(h)=o(h^\eta)$ as $h\to 0$. This proves the assertion for the first expression.

The second expression equals the first one by (4.4)″, the symmetry of μ about 1/2. Finally by using the change of variable discussed above we see that

$$\int_0^{h^{2/3}}|\mu(Q_h(x))|^2\leqslant(\rho/4)^{2N}\int_0^{h^{2/3}/\rho^{2N}}|\mu(Q_{h/\rho^{2N}}(x))|^2$$

$$\leqslant(\rho/4)^{2N}=o(h^\eta)$$

as $h\to 0$ also.

For simplicity we will use the notations

$$I(h)=\frac{1}{h^{1+\alpha}}\int_{\rho^2}^{\rho}|\mu(Q_h(x))|^2, \qquad J(h)=\frac{1}{h^{1+\alpha}}\int_0^{\rho^2}|\mu(Q_h(x))|^2,$$

$$K(h)=\frac{1}{h^{1+\alpha}}\int_0^{\rho^2}\mu(Q_h(x))\,\mu(Q_h(\rho^2-x)).$$

LEMMA 4.6. *Let ρ and μ be defined as in Theorem 4.4, then*

(i) $I(h)=(1/(2\rho^\alpha))(J(h/\rho)+K(h/\rho))$;

(ii) $J(h)=\sum_{n=1}^{\infty}(1/(4\rho^\alpha)^n)\,I(h/\rho^n)+E_1(h)$;

(iii) $K(h)=(1/(4\rho^\alpha))\,I(H/\rho)+(1/(4\rho^\alpha)^2)\,I(h/\rho^2)+E_2(h)$,

where $|E_i(h)|=o(h^\varepsilon)$, $i=1,2$ as $h\to 0$ for some $\varepsilon>0$.

We remark that the proof of this lemma can be represented in dynamics diagrams (module the error terms). It is given in the Appendix for reference.

Proof. (i) By using (4.4), and a change of variable, we have

$$\begin{aligned}
I(h)&=\frac{\rho}{4h^{1+\alpha}}\int_{\rho}^{1}|\mu(Q_{h/\rho}(x))+\mu(Q_{h/\rho}(x-\rho))|^2 \qquad \text{(by (4.4))}\\
&=\frac{\rho}{4h^{1+\alpha}}\int_0^{\rho^2}|\mu(Q_{h/\rho}(x))+\mu(Q_{h/\rho}(\rho^2-x))|^2 \qquad \text{(by 4.4)'')}\\
&=\frac{\rho}{4h^{1+\alpha}}\left(2\int_0^{\rho^2}|\mu(Q_{h/\rho}(x))|^2+2\int_0^{\rho^2}\mu(Q_{h/\rho}(x))\,\mu(Q_{h/\rho}(\rho^2-x))\right)\\
&=\frac{1}{2\rho^\alpha}(J(h/\rho)+K(h/\rho)).
\end{aligned}$$

To prove (ii), we first observe that

$$\begin{aligned}
&\frac{1}{h^{1+\alpha}}\int_{\rho^3}^{\rho^2}|\mu(Q_h(x))|^2\\
&\quad=\frac{1}{h^{1+\alpha}}\left(\int_{\rho^3}^{\rho^2-h}+\int_{\rho^2-h}^{\rho^2}\right)|\mu(Q_h(x))|^2\\
&\quad=\frac{1}{h^{1+\alpha}}\left(\frac{\rho}{4}\int_{\rho^2}^{\rho-h/\rho}|\mu(Q_{h/\rho}(x))|^2+\int_{\rho^2-h}^{\rho^2}|\mu(Q_h(x))|^2\right)\\
&\quad=\frac{1}{4\rho^\alpha}I(h/\rho)+e_1(h/\rho),
\end{aligned}$$

where $e_1(h/\rho)$ is defined in the obvious way. For $n \geqslant 2$, $0 < h^{2/3} < \rho^{n+2}$,

$$\begin{aligned}
&\frac{1}{h^{1+\alpha}} \int_{\rho^{n+1}}^{\rho^n} |\mu(Q_h(x))|^2 \\
&= \left(\frac{\rho}{4}\right)^{n-2} \frac{1}{h^{1+\alpha}} \int_{\rho^3}^{\rho^2} |\mu(Q_{h/\rho^{n-2}}(x))|^2 \\
&= \frac{1}{(4\rho^\alpha)^{n-2}} \left(\frac{1}{(4\rho^\alpha)} I(h/\rho^{n-1}) + e_1(h/\rho^{n-1})\right).
\end{aligned} \tag{4.6}$$

Let N be the largest number so that $0 < h/\rho^{3N} < \rho^2$ as in Lemma 4.5, then

$$\begin{aligned}
J(h) &= \frac{1}{h^{1+\alpha}} \left(\int_0^{\rho^{2N+1}} + \sum_{n=2}^{2N} \int_{\rho^{n+1}}^{\rho^n}\right) |\mu(Q_h(x))|^2 \\
&= e_2(h) + \sum_{n=2}^{2N} \left(\frac{1}{(4\rho^\alpha)^{n-1}} I(h/\rho^{n-1}) + \frac{1}{(4\rho^\alpha)^{n-2}} e_1(h/\rho^{n-1})\right) \\
&= \sum_{n=1}^{\infty} \frac{1}{(4\rho^\alpha)^n} I(h/\rho^n) + E_1(h),
\end{aligned}$$

where

$$E_1(h) = e_2(h) + \sum_{n=2}^{2N} \frac{1}{(4\rho^\alpha)^{n-2}} e_1(h/\rho^{n-1}) + \sum_{n=2N+1}^{\infty} \frac{1}{(4\rho^\alpha)^{n-1}} I(h/\rho^{n-1}).$$

The first two terms are of order $o(h^\varepsilon)$ for $0 < \varepsilon < \eta - 2$ by Lemma 4.5. The last term is also of order $o(h^\varepsilon)$ for some $\varepsilon > 0$ by noting that $0 < \alpha \leqslant 1$, $4\rho^\alpha > 4\rho > 2$, so that it is dominated by

$$\begin{aligned}
&\sum_{n=2N}^{3N} \frac{1}{(4\rho)^n (h/\rho^n)^2} + \sum_{n=3N+1}^{\infty} \frac{1}{2^n} \\
&= h^{-2} \sum_{n=2N}^{3N} \frac{\rho^n}{4n} + 2^{-3N} < C\rho^{-6N} \left(\frac{\rho}{4}\right)^{2N} + 2^{-3N} \\
&= C(2\rho)^{-4N} + 2^{-3N}.
\end{aligned}$$

For (iii), we write

$$\begin{aligned}
K(h) &= \frac{1}{h^{1+\alpha}} \left(\int_0^{\rho^4} + \int_{\rho^4}^{\rho^3} + \int_{\rho^3}^{\rho^2}\right) \mu(Q_h(x))\, \mu(Q_h(\rho^2 - x)) \\
&= T_1 + T_2 + T_3, \qquad \text{say.}
\end{aligned}$$

By applying the previous technique and the symmetry property of μ about 1/2 (i.e., (4.4)″), we have

$$T_2=\left(\frac{\rho}{4}\right)^2\frac{1}{h^{1+\alpha}}\int_{\rho^2}^{\rho}\mu(Q_{h/\rho^2}(x))\,\mu(Q_{h/\rho^2}(1-x))$$
$$=\left(\frac{\rho}{4}\right)^2\frac{1}{h^{1+\alpha}}\int_{\rho^2}^{\rho}\mu(Q_{h/\rho^2}(x))^2=\frac{1}{(4\rho^\alpha)^2}I(h/\rho^2)+e_3(h),$$

where $e_3(h)=o(h^\varepsilon)$ as $\varepsilon\to 0$, by Lemma 4.5. By a change of variable of $y=\rho^2-x$ for the x in T_3, we see that $T_1=T_3$. Also

$$T_1=\frac{\rho}{4h^{1+\alpha}}\int_0^{\rho^3}\mu(Q_{h/\rho}(x))\,\mu(Q_{h/\rho}(\rho-x))+e_4(h)$$
$$=\frac{\rho^2}{4^2h^{1+\alpha}}\int_0^{\rho^2}\mu(Q_{h/\rho^2}(x))\{\mu(Q_{h/\rho^2}(1-x))+\mu(Q_{h/\rho^2}(1-x-\rho))\}+e_4(h)$$
$$=\frac{1}{(4\rho^\alpha)^2}\,(I(h/\rho^2)+K(h/\rho^2))+e_4(h),$$

where $e_4(h)=o(h^\varepsilon)$ as $h\to 0$. (The second identity makes use of (4.4) applied to $\rho^2<\rho-x<\rho$ for $0<x<\rho^3$; the last equality follows from (4.4)′ and $1-\rho=\rho^2$.)

Finally combining the above identity and (i), we have

$$K(h)=\frac{1}{(4\rho^\alpha)^2}\,(I(h/\rho^2)+2J(h/\rho^2)+2K(h/\rho^2))+E_2(h)$$
$$=\frac{1}{(4\rho^\alpha)}I(h/\rho)+\frac{1}{(4\rho^\alpha)^2}I(h/\rho^2)+E_2(h). \tag{4.7}$$

Proof of Theorem 4.4. Write $c=4\rho^\alpha(>2)$. By Lemma 4.6, we have

$$I(h)=2c^{-1}(J(h/\rho)+K(h/\rho))$$
$$=2c^{-1}\left(\sum_{n=1}^{\infty}c^{-n}I(h/\rho^{n+1})+c^{-1}I(h/\rho)+c^{-2}I(h/\rho^2)\right)+E(h),$$

where $E(h)$ is defined in an obvious way, and is of order $o(h^\varepsilon)$ as $h\to 0$.

By letting $x=-\ln h$, $f(x)=I(e^{-x})$, $S(x)=\int_{-\infty}^{-x}f(x+y)\,dv(y)+E(e^{-x})$, we can rewrite the above equation as

$$f(x)=\int_{-\infty}^{0}f(x+y)\,dv(y)+E(e^{-x})$$
$$=\int_0^x f(x-y)\,d\tilde{v}(y)+S(x),\qquad x\geqslant 0,$$

where the definition of v and $\tilde{v}$ are self-explained. Note that S is not identically zero, bounded with $|S(x)| = o(e^{-\varepsilon x})$ as $x \to 0$. Also note that the weight of $\tilde{v}$ is given by

$$2c^{-1}\left(\sum_{n=1}^{\infty} c^{-n} + c^{-1} + c^{-2}\right) = \frac{4c^2 - 2}{c^3(c-1)},$$

which equals 1 if and only if $c^3 - 2c^2 - 2c + 2 = 0$. The equation has three roots but only one satisfies $c > 2$. It follows from the hypothesis of α that $\tilde{v}$ is a probability measure. Moreover $\int_0^\infty y \, d\tilde{v}(y) < \infty$. Hence Theorem 4.1(ii) implies that f is equal to a non-zero multiplicative periodic function p_1 asymptotically at ∞, i.e.,

$$\lim_{h \to 0} (I(h) - p_1(h)) = 0.$$

Observe that

$$\frac{1}{h^{1+\alpha}} \int_0^1 |\mu(Q_h(x))|^2 = 2J(h) + I(h),$$

and the relationship of $I(h)$ and $J(h)$ in Lemma 4.6, we have

$$\lim_{h \to 0} \left(\frac{1}{h^{1+\alpha}} \int_0^1 |\mu(Q_h(x))|^2 - p(h)\right) = 0,$$

and the proof of (i) is complete. Part (ii) follows from (i) and Theorem 3.2.

We remark that we are not able to find a general expression of the m.q.v. indices of the self-similar measures μ_ρ where ρ^{-1} are P.V. numbers, in particular, for the next most important P.V. number: the smallest of such a number, which is a root of $x^3 - x - 1 = 0$ [G]. Also there is a well known open problem in this direction: determine $1/2 < \rho < 1$ so that μ_ρ is absolutely continuous; the problem is a consequence of characterizing $1/2 < \rho < 1$ so that μ_ρ has m.q.v. index 1.

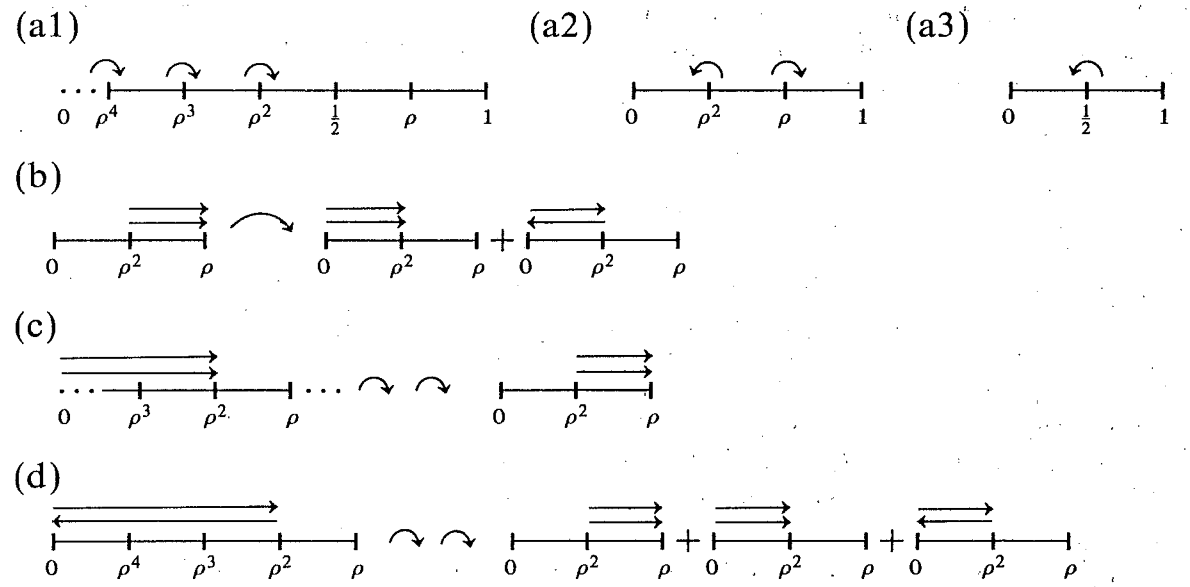
To conclude this section, we let $\{R_n\}_{n=1}^{\infty}$ be the sequence of Rademacher functions and let

$$R(x) = \sum_{n=1}^{\infty} 2^{-\beta n} R_n, \qquad x \in [0, 1].$$

The distribution function F of R is partially known (F can be identified with μ_ρ with $\rho = 2^{-\beta}$) from Theorem 4.3, Theorem 4.4, and their remarks. If the distribution function F is absolutely continuous and $F' \in L^p$ for some $p > 1$, then the Hausdorff dimension of the graph of R is $2 - \beta$ [HL1, PU], and the Hausdorff dimension of the level set is $1 - \beta$ a.e. [HL2]. By using a dynamic argument, Przytycki and Urbanski [PU] proved a more striking result: if 2^β is a P.V. number, then the Hausdorff dimension of the graph is less than $2-\beta$. This is in contrast to the result that the "box" dimension of graphs of this type (including the Weierstrass function) is $2-\beta$, and the general belief that the same is true for the Hausdorff dimension. In connection to Theorem 4.4, it is natural to ask: If F has m.q.v. index α, what is the exact Hausdorff dimension of R in terms of α and β?

Appendix

In the following we will summarize the self-similar property (4.4)' and the proof of Lemma 4.6 into the following symbolic dynamic diagrams.



Diagrams (a1) and (a2) represent the self-similar property applied to the intervals $[\rho^{n+1}, \rho^n]$, $n \geqslant 1$ (see (4.4) and (4.4)'). Diagram (a3) is the reflection of the interval $[\overrightarrow{1/2, 1}]$ to $[\overleftarrow{0, 1/2}]$ (opposite direction) due to the symmetry of μ with respect to 1/2.

In (b), (c), and (d), the pairs of arrows represents the regions of the quadratic integrals associated with the directions; e.g., the first one in (b) and (d) means

$$\int_{\rho^2}^{\rho} |\mu(Q_h(x))|^2 \qquad \text{and} \qquad \int_0^{\rho^2} \mu(Q_h(x))\, \mu(Q_h(\rho^2 - x)),$$

respectively.

Diagram (b) represents the change of the regions of integration (with direction) of Lemma 4.6(i) after applying (a2) and (a3). The application of (a2) (and also (a1)) produces a factor of $1/(4\rho)^\alpha$ to the integral, and the variable h changes to h/ρ.

Diagram (c) represents Lemma 4.6(ii), applying (a1) to each region $[\rho^{n+1}, \rho^n]$, $n \geq 2$, repeatedly to land on $[\rho^2, \rho]$. The error terms are omitted.

Diagram (d) represents Lemma 4.6(iii) (actually (4.7)) by applying (a1) twice to the interval $[\rho^4, \rho^3]$, $n \geq 1$, to produce the first summand (T_2 in the proof), and applying (a1) and (a3) to $[0, \rho^4]$ (the same for $[\rho^3, \rho^2]$) to produce the last two summands of (d) (T_1 in the proof).

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